Weak form of bond-associated peridynamics for finite elastic deformation and rupture in rubber-like materials

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Outline

- Existing models
- Peridynamic (PD) theory
- Bond-associated deformation gradient
- Weak form of PD
- Numerical results
- Remarks

Existing models

FEA with criteria for crack initiation and propagation

- J-integral criteria (Hocine et al., 1998, 2002, 2003),
- Local energy release rate criteria (Mzabi et al., 2011)
- Strain energy density (SED) criteria (Hamdi et al., 2007)
- Local (or average) SED criteria (Berto, 2015)
- Tearing energy (Rivlin and Thomas,1953 and Pidaparti et al., 1990)
- Maximum principal stretch and stress (Hamdi et al., 2007)
- Effective stretch (Ayatollahi et al., 2016)

Such criteria are based on local stress and strain fields

Predictions are significantly influenced by the mesh size

FEA coupled with a phase-field approach (Talamini et al., 2018)

- Length scale to predict the rupture process
- Phase-field is driven only by the internal energy of polymer

It successfully describes damage initiation, propagation, and rupture

Existing peridynamic models

PD form of strain energy density functions

- Tearing of a rubber sheet (Silling and Bobaru (2005)
- Uniform stretch of a rubber sheet w/o failure (Bang and Madenci, 2017)
- 3D simulation of hyperelastic materials w/o failure (Xu et al., 2018)
- Visco-hyperelastic deformation of polymers (Huang et al., 2019)

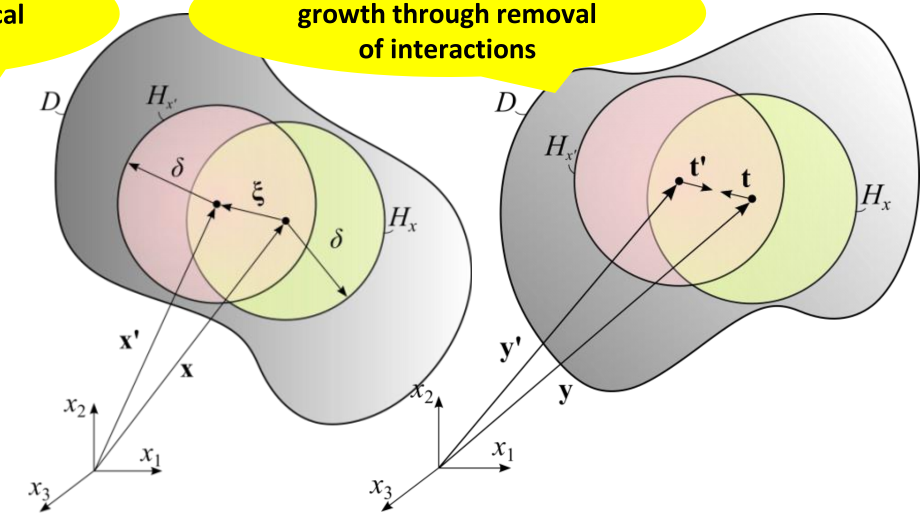
Classical form of strain energy density functions

- Arruda-Boyce model w/o failure (Henke, 2013)
- Neo-Hookean model w/o failure (Waxman and Guven, 2020)

Peridynamic theory

As horizon approaches zero, it recovers local theory

Failure initiation and growth through removal of interactions



$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}(\mathbf{x},t) = \rho \ddot{\mathbf{u}}(\mathbf{x},t)$$

$$\int_{H_{\mathbf{x}}} (\mathbf{t} - \mathbf{t}')\, dV + \mathbf{b}(\mathbf{x},t) = \rho \ddot{\mathbf{u}}(\mathbf{x},t)$$

$$\mathbf{t}(\mathbf{x}) = w(|\xi|)\, \mathbf{P}\mathbf{K}^{-1}\xi$$

$$\mathbf{K} = \int_{H_{\mathbf{x}}} w(|\,\xi\,|)(\xi \otimes \xi\,)\, dV_{\mathbf{x}'}$$

$$\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}$$

No spatial derivatives
Internal length parameter
Valid at discontinuities

Boundary conditions
- Displacement constraints - natural
- Nonzero tractions - body forces
- Zero tractions - Lagrange mult.

Silling S. A., 2000, "Reformulation of elasticity theory for discontinuities and long-range forces," JMPS, 48 : 175 -209.
Silling et al. , 2007, "Peridynamics states and constitutive modeling," J. Elasticity, 88 : 151-184

PD form of deformation gradient

Silling et al. (2007) - “*reduction*” process

$$\mathbf{y}' = \mathbf{y} + \left(\nabla \mathbf{y}(\mathbf{x})\right)\boldsymbol{\xi} + R(1, \mathbf{x})$$

$$\mathbf{y}' - \mathbf{y} = \mathbf{F}(\mathbf{x})\boldsymbol{\xi}$$

$$\left(\mathbf{y}' - \mathbf{y}\right) \otimes \boldsymbol{\xi} = \mathbf{F}(\mathbf{x})\left(\boldsymbol{\xi} \otimes \boldsymbol{\xi}\right)$$

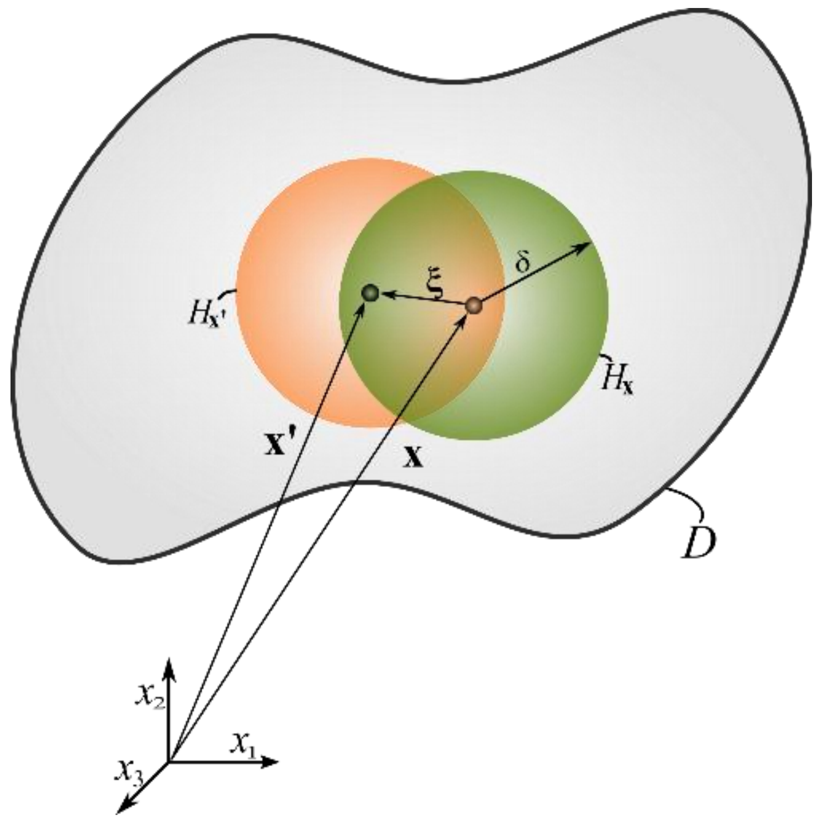
$$\int_{H_{\mathbf{x}}} w(|\boldsymbol{\xi}|)\left(\mathbf{y}' - \mathbf{y}\right) \otimes \boldsymbol{\xi}\, dV_{\mathbf{x}'} = \mathbf{F}(\mathbf{x})\mathbf{K}(\mathbf{x})$$

$$\mathbf{K}(\mathbf{x}) = \int_{H_{\mathbf{x}}} w(|\boldsymbol{\xi}|)\left(\boldsymbol{\xi} \otimes \boldsymbol{\xi}\right) dV_{\mathbf{x}'}$$

$$\mathbf{F}(\mathbf{x}) = \left(\int_{H_{\mathbf{x}}} w(|\boldsymbol{\xi}|)\left(\mathbf{y}' - \mathbf{y}\right) \otimes \boldsymbol{\xi}\, dV_{\mathbf{x}'}\right)\mathbf{K}^{-1}$$

$$\mathbf{F}(\mathbf{x}) = \int_{H_{\mathbf{x}}} \left(\mathbf{y}' - \mathbf{y}\right) \otimes \mathbf{g}\, dV_{\mathbf{x}'}$$

$$\mathbf{g} = w(|\boldsymbol{\xi}|)\mathbf{K}^{-1}\boldsymbol{\xi}$$



Silling et al. , 2007, “Peridynamics states and constitutive modeling,” J. Elasticity, 88 : 151-184

F(x) at two different states

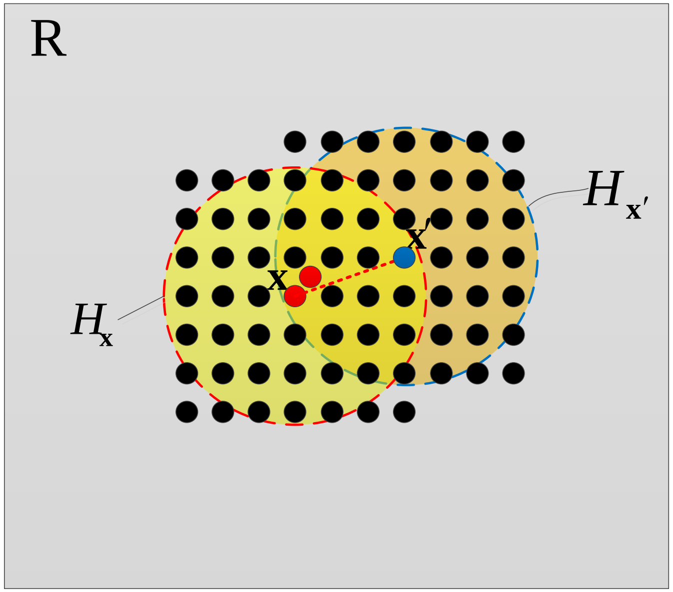
$$\mathbf{y}_{\text{new}}(\mathbf{x}) = \mathbf{y}_{\text{old}}(\mathbf{x}) + \mathbf{u}_{\text{d}}(\mathbf{x})$$

$$\mathbf{y}_{\text{new}}(\mathbf{x}') = \mathbf{y}_{\text{old}}(\mathbf{x}')$$

$$\mathbf{F}_{\text{old}}(\mathbf{x}) = \left(\int_{H_{\mathbf{x}}} w(|\xi|)(\mathbf{y}_{\text{old}}(\mathbf{x}') - \mathbf{y}_{\text{old}}(\mathbf{x})) \otimes (\mathbf{x}' - \mathbf{x}) dV_{\mathbf{x}'} \right) \mathbf{K}^{-1}(\mathbf{x})$$

$$\mathbf{F}_{\text{new}}(\mathbf{x}) = \left(\int_{H_{\mathbf{x}}} w(|\xi|)\left(\mathbf{y}_{\text{new}}(\mathbf{x}') - \mathbf{y}_{\text{new}}(\mathbf{x})\right) \otimes (\mathbf{x}' - \mathbf{x}) dV_{\mathbf{x}'} \right) \mathbf{K}^{-1}(\mathbf{x})$$

$$\mathbf{F}_{\text{new}}(\mathbf{x}) = \left(\int_{H_{\mathbf{x}}} w(|\xi|)\left(\mathbf{y}_{\text{new}}(\mathbf{x}') - \left(\mathbf{y}_{\text{old}}(\mathbf{x}) + \mathbf{u}_{\text{d}}(\mathbf{x})\right)\right) \otimes (\mathbf{x}' - \mathbf{x}) dV_{\mathbf{x}'} \right) \mathbf{K}^{-1}(\mathbf{x})$$



Uniform discretization and a spherically symmetric influence function

$$\mathbf{F}_{\text{new}}(\mathbf{x}) = \mathbf{F}_{\text{old}}(\mathbf{x}) - \mathbf{u}_{\text{d}}(\mathbf{x}) \otimes \underbrace{\left(\int_{H_{\mathbf{x}}} w(|\xi|)(\mathbf{x}' - \mathbf{x}) dV_{\mathbf{x}'} \right)}_{=0} \mathbf{K}^{-1}(\mathbf{x})$$

Zero deformation energy due to a displacement variation

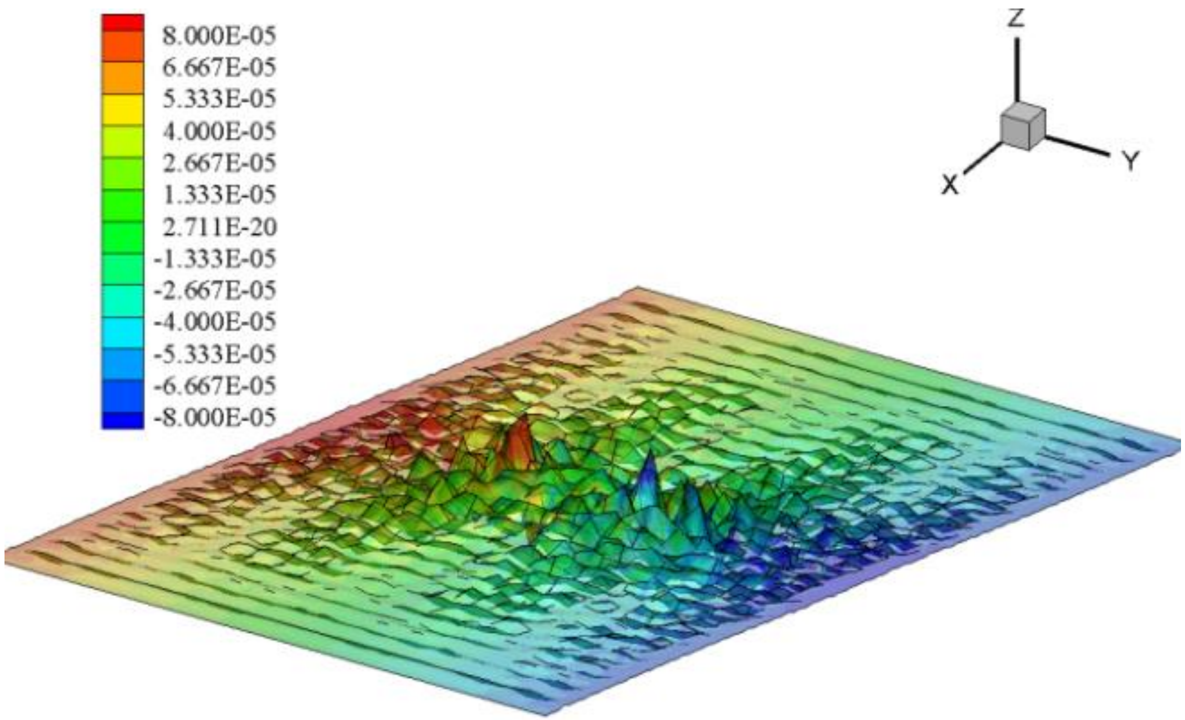
Zero-energy deformation mode

Chen H and Spencer BW, Peridynamic Bond-Associated Correspondence Model: Stability and Convergence Properties. IJNME, 2018; 117, 713-7273

Zero-energy deformation modes

- Numerical instability/oscillation - in regions of steep deformation gradient such as a crack

- Stabilization techniques - artificial

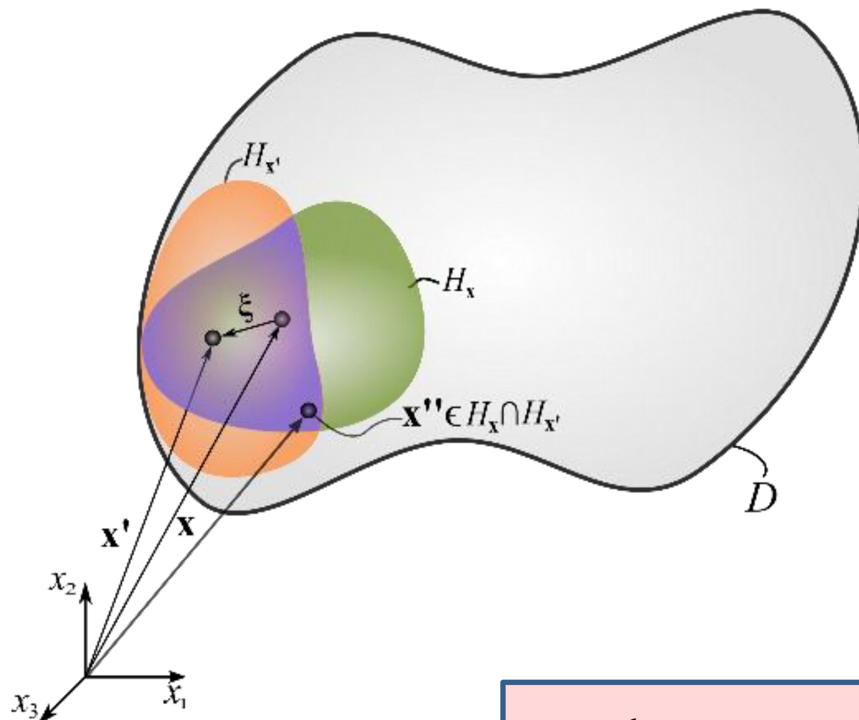


Numerical stabilization techniques

- Enhanced integration methods
- Adding an extra force vector state (damping term)
- Suitable influence function
- Averaging/Smoothing displacement field
- Higher-order approximation deformation gradient
- Stress point method
- Sub-horizon average method
- Variable correction method at bond-level
- Bond-associated horizon
- **Bond-associated deformation gradient**
  - Disturbs the radial symmetry and removes the zero-energy mode

Chen H, Bond-associated deformation gradients for peridynamic correspondence model. Mechanics Research Comm., 2018, 90:34-41.
Chen H and Spencer BW, Peridynamic Bond-Associated Correspondence Model: Stability and Convergence Properties. IJNME, 2018; 117, 713-7273

Bond-associated deformation gradient



$$\mathbf{F}_{\xi}(\mathbf{x}) = \int_{H_{\mathbf{x}} \cap H_{\mathbf{x}'}} \left(\mathbf{y}'' - \mathbf{y}\right) \otimes \mathbf{g}_{\xi} \, dV_{\mathbf{x}''}$$

$$\mathbf{g}_{\xi} = \begin{Bmatrix} g_1^{100}\left(\xi; w(|\xi|)\right) \\ g_1^{010}\left(\xi; w(|\xi|)\right) \\ g_1^{001}\left(\xi; w(|\xi|)\right) \end{Bmatrix}$$

Chen H, Bond-associated deformation gradients for peridynamic correspondence model. Mechanics Research Comm., 2018, 90:34-41.

Chen H and Spencer BW, Peridynamic Bond-Associated Correspondence Model: Stability and Convergence Properties. IJNME, 2018; 117, 713-7273

Madenci E, Barut A, Dorduncu M, Peridynamic Differential Operator for Numerical Analysis, Springer, 2019.

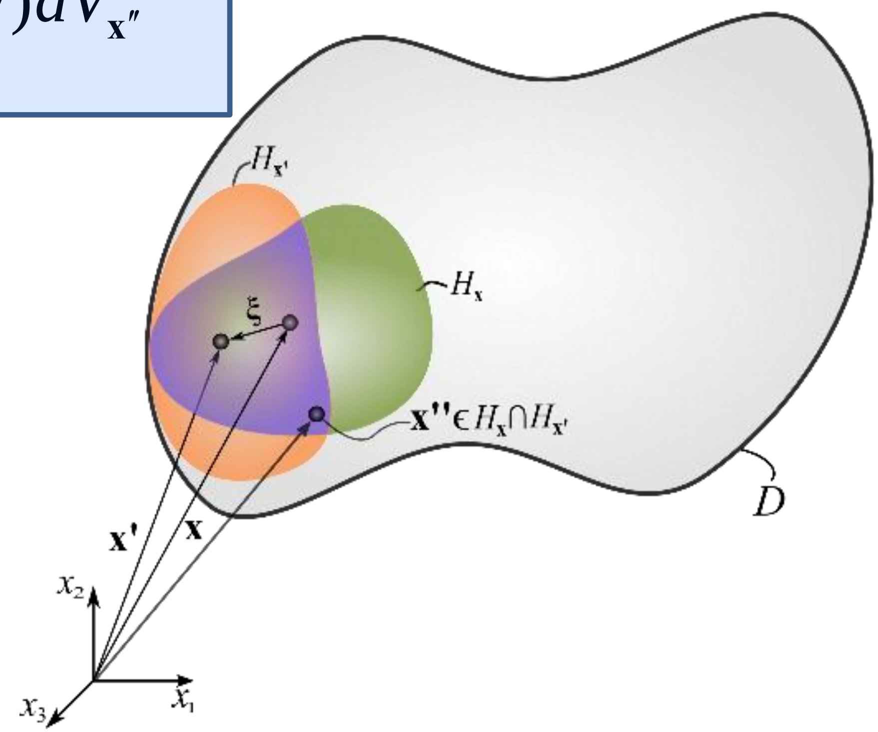
Bond strain energy density

$$\delta W_{\xi}^{PD}(\mathbf{x}) = \frac{1}{2} \int\limits_{H_{\mathbf{x}} \cap H_{\mathbf{x}'}} \left(\mathbf{t}_{\xi}(\mathbf{x}) - \mathbf{t}_{\xi}(\mathbf{x}'')\right) \bullet \delta\left(\mathbf{y}(\mathbf{x}'') - \mathbf{y}(\mathbf{x})\right) dV_{\mathbf{x}''}$$

$$\delta W_{\xi}^{CCM}(\mathbf{x}) = \int\limits_{H_{\mathbf{x}} \cap H_{\mathbf{x}'}} (\mathbf{P}_{\xi}\mathbf{g}_{\xi})^{T} \delta(\mathbf{y}'' - \mathbf{y}) dV_{\mathbf{x}''}$$

$$\delta W_{\xi}^{CCM}(\mathbf{x}) = \mathrm{tr}(\mathbf{P}_{\xi}^{T} \, \delta\mathbf{F}_{\xi})$$

$$\mathbf{P}_{\xi} = \frac{\partial \Psi(\mathbf{F}_{\xi})}{\partial \mathbf{F}_{\xi}}$$



Hu YL, Chen H, Spencer BW, Madenci, E., Thermomechanical peridynamic analysis with non-uniform discretization, Engineering Fracture Mechanics, 2018, 197: 92-113.

Equating PD and classical SED

$$\delta W_{\xi}^{PD}(\mathbf{x}) = \phi_{\xi}(\mathbf{x},\mathbf{x}')\,\delta W_{\xi}^{CCM}(\mathbf{x})$$

Assumption

$$\phi_{\xi}(\mathbf{x},\mathbf{x}') = \frac{\int_{H_{\mathbf{x}} \cap H_{\mathbf{x}'}} dV_{\mathbf{x}''}}{\int_{H_{\mathbf{x}}} dV_{\mathbf{x}''}}$$

$$\mathbf{L}^{\mathrm{PD}}(\mathbf{x},t) = \int_{H_{\mathbf{x}}} \Big(\phi_{\xi}(\mathbf{x},\mathbf{x}')\mathbf{P}_{\xi}\,\mathbf{g}_{\xi}(\mathbf{x}) - \phi_{\xi}(\mathbf{x}',\mathbf{x})\mathbf{P}'_{\xi}\,\mathbf{g}_{\xi}(\mathbf{x}') \Big) dV_{\mathbf{x}'}$$

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Neo-Hookean material model

$$\Psi(\bar{I}_1, J) = \frac{\mu}{2}\left(\bar{I}_1 - 3\right) + \frac{K}{8}\left(J - \frac{1}{J}\right)^2$$

$$\bar{I}_1 = \frac{\mathrm{tr}\mathbf{C}}{J^{2/3}} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} \qquad J = \det\mathbf{F}$$

$$\mathbf{P} = \frac{\partial\Psi}{\partial\mathbf{F}} = \mu\left(\mathbf{F} - \frac{1}{3}\mathrm{tr}\mathbf{C}\mathbf{F}^{-T}\right)J^{-2/3} + \frac{K}{4}\left(J^2 - J^{-2}\right)\mathbf{F}^{-T}$$

$$\mathbf{P} = \mu\left(\mathbf{F} - \frac{1}{2}\mathrm{tr}\mathbf{C}\mathbf{F}^{-T}\right)J^{-1} + \frac{K}{4}\left(J^2 - J^{-2}\right)\mathbf{F}^{-T}$$ -- plane strain

$$\mathbf{P} = \mu\left(\mathbf{F} - \frac{1}{3}C_{33}\mathbf{F}^{-T}\right)J^{-2/3}$$ -- plane stress

B Talamini, Y Mao, L Anand Progressive damage and rupture in polymers Journal of the Mechanics and Physics of Solids 2018 111, 434-457

Weak form of Peridynamics

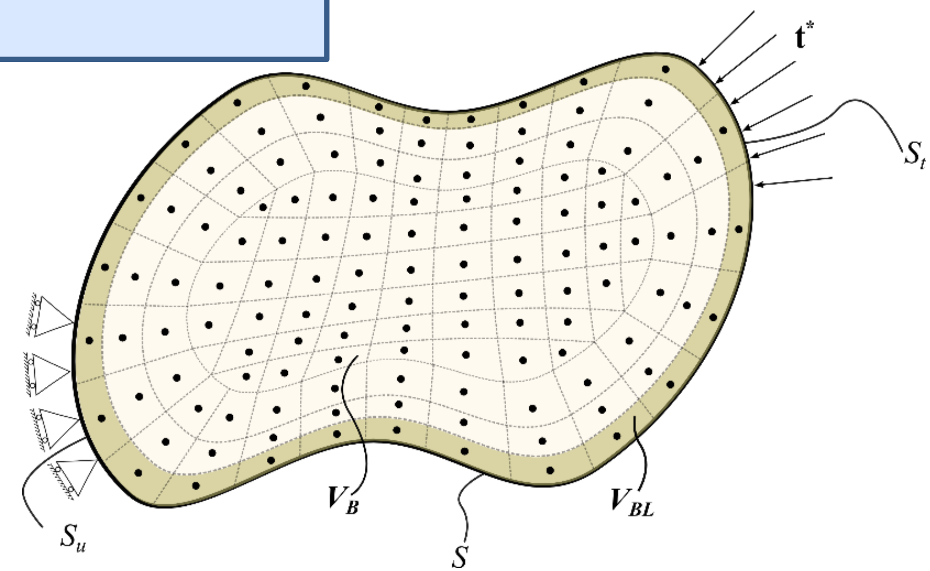
$$\delta U + \delta V = 0$$

$$\oint_{\Gamma} (\mathbf{P}^{\mathrm{PD}}\mathbf{n}) \bullet \delta\mathbf{u}\, dS - \int_{A} \mathbf{L}^{\mathrm{PD}} \bullet \delta\mathbf{u}\, dV - \int_{\Gamma_{\mathbf{t}}} \mathbf{t}^{*} \bullet \delta\mathbf{u}\, dS = 0$$

$$\int_{A} \mathbf{L}^{\mathrm{PD}} \cdot \delta\mathbf{u}\, dA \approx \delta\mathbf{V}^{T}\mathbf{Q}$$

$$\int_{\Gamma} (\mathbf{P}^{\mathrm{PD}}\mathbf{n}) \cdot \delta\mathbf{u}\, d\Gamma = \delta\mathbf{V}_{\mathbf{BL}}^{T}\mathbf{q}_{\mathbf{BL}}$$

$$\int_{\Gamma_{\mathbf{t}}} \mathbf{t}^{*} \bullet \delta\mathbf{u}\, d\Gamma = \delta\mathbf{V}_{\mathbf{t}}^{T}\mathbf{q}_{\mathbf{t}}^{*}$$



$$\mathbf{V} = \left\{ \mathbf{V}_{\mathbf{B}} \quad \mathbf{V}_{\mathbf{BL}} \right\}^{T} = \left\{ \mathbf{V}_{\mathbf{B}} \quad \mathbf{V}_{\mathbf{t}} \quad \mathbf{V}_{\mathbf{u}} \right\}^{T}$$

$$\mathbf{Q} = \left\{ \mathbf{Q}_{\mathbf{B}} \quad \mathbf{Q}_{\mathbf{BL}} \right\}^{T} = \left\{ \mathbf{Q}_{\mathbf{B}} \quad \mathbf{Q}_{\mathbf{t}} \quad \mathbf{Q}_{\mathbf{u}} \right\}^{T}$$

$$\mathbf{q}_{\mathbf{BL}} = \left\{ \mathbf{q}_{\mathbf{t}} \quad \mathbf{q}_{\mathbf{u}} \right\}^{T}$$

$$\left\{ \mathbf{Q}_{\mathbf{B}} \quad \left(\mathbf{Q}_{\mathbf{t}} - \mathbf{q}_{\mathbf{t}} + \mathbf{q}_{\mathbf{t}}^{*} \right) \quad \left(\mathbf{Q}_{\mathbf{u}} - \mathbf{q}_{\mathbf{u}} \right) \right\} \begin{Bmatrix} \delta\mathbf{V}_{\mathbf{B}} \\ \delta\mathbf{V}_{\mathbf{t}} \\ \delta\mathbf{V}_{\mathbf{u}} \end{Bmatrix} = 0$$

Madenci, E., Dorduncu, M., Barut, A. and Phan, N., 2018, “Weak form of peridynamics for nonlocal essential and natural boundary conditions,” CMAME, Vol. 337, pp. 598-631.

Nonlinear governing equations

$$\mathbf{R}\left(\mathbf{V};\mathbf{f}_{\mathbf{t}}^{*}\right)=\begin{Bmatrix}\mathbf{Q}_{\mathbf{B}}\\ \mathbf{Q}_{\mathbf{t}}-\mathbf{q}_{\mathbf{t}}+\mathbf{q}_{\mathbf{t}}^{*}\\ \mathbf{Q}_{\mathbf{u}}-\mathbf{q}_{\mathbf{u}}\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{Bmatrix}$$

$$\mathbf{R}\left(\mathbf{V};\mathbf{f}_{\mathbf{t}}^{*}\right)=\begin{Bmatrix}\mathbf{Q}_{\mathbf{B}}\\ \mathbf{Q}_{\mathbf{t}}-\mathbf{q}_{\mathbf{t}}+\mathbf{q}_{\mathbf{t}}^{*}\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\end{Bmatrix}\quad \text{for}\ \ \mathbf{V}_{\mathbf{u}}=\mathbf{V}_{\mathbf{u}}^{*}$$

$$\left(\frac{\partial \mathbf{R}}{\partial \mathbf{V}}\right)^{(n)}\Delta\mathbf{V}^{(n+1)}=-\mathbf{R}(\mathbf{V}^{n};\mathbf{q}_{\mathbf{t}}^{*})-\Delta\mathbf{q}_{\mathbf{t}}^{*}$$

$$\mathbf{R}(\mathbf{V}^{0};\mathbf{f}_{\mathbf{t}}^{*})=\mathbf{0}\qquad\qquad \mathbf{V}^{(n+1)}=\mathbf{V}^{(n)}+\Delta\mathbf{V}^{(n+1)}$$

Intel direct solver
PARDISO