Limit of Nonlocal Dynamics.

Theorem

Convergence of nonlocal dynamics. L. 2016 Bond based, Jha and L. 2019 State Based double well cohesive models

For each ϵ we prescribe the same initial data $u_0(x)$ and $v_0(x)$. Then the cohesive evolutions u^ϵ converge in mean square uniformly in time to a limit evolution u^0

The limit evolution

Theorem

The limit dynamics has bounded *LEFM* **energy. L. 2014, 2016 Bond based, Jha and Lipton, State Based**

There exists a constant C depending only on T bounding the LEFM energy,

$$\int_D 2\overline{\mu}|\mathcal{E}u^0(t)|^2 + \overline{\lambda}|\mathrm{div}\, u^0(t)|^2\, dx + \mathcal{G}_c \times Area(Crack(t)) \leq C, \;\; d = 2,3, \quad (10)$$

for $0 \leq t \leq T$.

Elastodynamics away from the failure zone

Theorem (2016 L. Bond based; Jha and L. State based 2019)

Fix $\delta > 0$ and consider any set $D' \subset D$ for which points x in D' do not belong to the failute zone for every evolution $u^{\epsilon}(t,x)$
Then the limit evolution $u^0(t,x)$ evolves elastodynamically on D' and is governed by the balance of linear momentum expressed by the Navier Lamé equations on the domain $[0,T] \times D'$ given by

$$\rho u^0_{tt} = \mathrm{div}\sigma + b, \text{ on } [0,T] \times D', \tag{11}$$

Elastodynamics away from the damage zone

The stress tensor σ is given by,

$$\sigma = \mathrm{I}\bar{\lambda}\mathrm{div}\, \boldsymbol{u}^0 + 2\bar{\mu}\mathcal{E}\, \boldsymbol{u}^0, \tag{12}$$

The interaction between the crack and intact material as horizon goes to zero

Traction boundary conditions, symmetric notched plate in tension

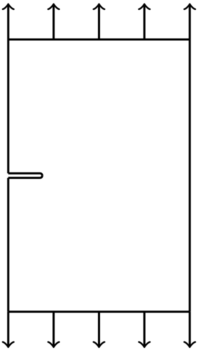


Figure: **Single-edge-notch**

Simulation for the nonlocal model

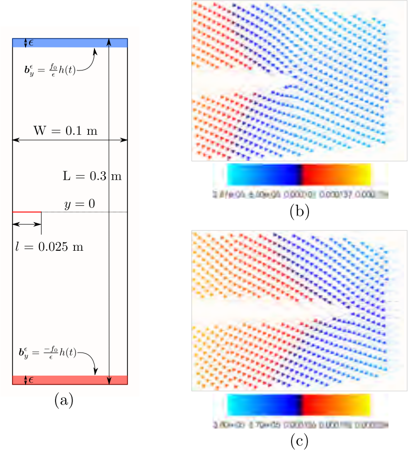


Figure: **Displacement in the domain at time $t = 460\ \mu$s and $t = 520\ \mu$s for horizon $\epsilon = 0.625$ mm.**

No nonlocal forces across centerline

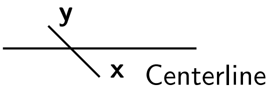
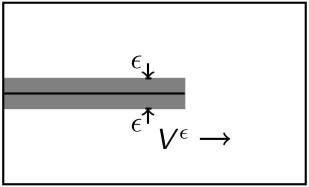


Figure: **No interaction between x and y on either side of center line.**



Figure

The limit of vanishing horizon:

As $\epsilon \to 0$ the double well nonlocal evolution converges to the dynamic brittle fracture model given by:

- Balance of linear momentum described by the linear elastic wave equation away from the crack.
- Zero traction on the crack lips.

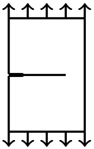


Figure: **Single-edge-notch**

Interaction between centerline line tip and intact material for non local model

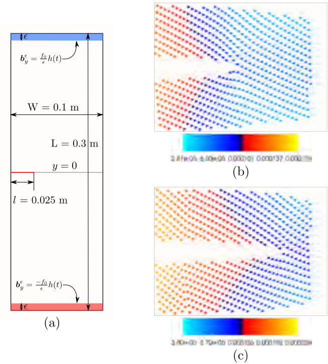


Figure: **Displacement in the domain at time $t = 460\ \mu$s and $t = 520\ \mu$s for horizon $\epsilon = 0.625$ mm.**

Interaction between failure zone and intact material: J integral

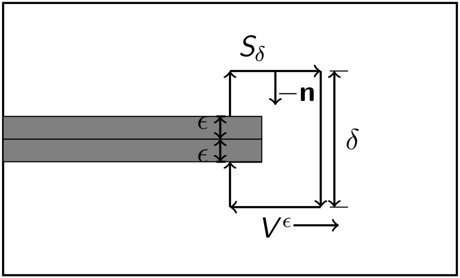


Figure: **The contour S_δ surrounding the failure zone (in gray) moving with the failure zone.**

Energy flux through S_δ

The kinetic energy and stress power density on S_δ are defined by

- $$T^\epsilon = \rho|\boldsymbol{u}^\epsilon(\boldsymbol{x},t)|^2/2$$
- $$W^\epsilon(\boldsymbol{x}) = \int_{\mathcal{H}_\epsilon(\boldsymbol{x})} |\boldsymbol{y}-\boldsymbol{x}|\mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t)))\,d\boldsymbol{y}$$

Energy flux into failure zone tip through S_δ

The energy flux into failure zone tip through S_δ is given by the J^ϵ integral

$$J^\epsilon(S_\delta(t)) = -\int_{S_\delta(t)} (T^\epsilon + W^\epsilon)V^\epsilon \boldsymbol{e}^1 \cdot \boldsymbol{n}\, ds + E^\epsilon(S_\delta(t)),$$

and

$$E^\epsilon(S_\delta(t)) = \int_{A_\delta(t)} \int_{\mathcal{H}_\epsilon(\boldsymbol{x})\cap Q_\delta(t)} \partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon))\boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}} \cdot (\dot{\boldsymbol{u}}^\epsilon(\boldsymbol{x}) + \dot{\boldsymbol{u}}^\epsilon(\boldsymbol{y}))\, d\boldsymbol{y} d\boldsymbol{x},$$

where $\boldsymbol{n}$ is the unit normal pointing out of the interior of $S_\delta(t)$ ($Int\{S_\delta\}$) and $A_\delta(t)$ is the part of D exterior to $Int\{S_\delta\}$. Other formulations for peridynamic J integral given in Silling and Lehoucq (2010), J. Foster, Silling, and Chen (2011), Hu, Ha, Bobaru, and Silling (2012).

The limit of the nonlocal J integral delivers the local J integral

- For remote loads far away from the crack tip the rate of change in internal energy as $\epsilon \rightarrow$ of a neighborhood of the crack tip is

$$\lim_{\epsilon\rightarrow 0} J^{\epsilon} = J = \frac{1+\nu}{E}\frac{V^3}{c_s^2 D}\alpha_t K_I^2(t) + O(\delta), \tag{13}$$

  Here V is crack tip velocity

- From energy balance the fracture toughness is equal to elastic energy flowing into the crack and the standard kinetic relation for crack tip motion is

$$\mathcal{G}_c = J/V$$

  providing an implicit equation for crack velocity.

A second way to get kinetic relation

- The double well nonlocal energy is defined everywhere, the hallmark of peridynamic modeling
- Calculate rate of change of energy inside domain containing crack tip *not excluding it as done with J integral approach*.
- More information from nonlocal model can be obtained. See Jha & Lipton, arXiv.org ArXiv:1908.07589, preprint 2020.

Interaction between failure zone and intact material, *We proceed another way:* **a completely nonlocal calculation at fixed ϵ**

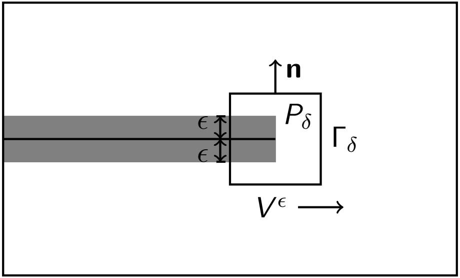


Figure: **Contour Γ_δ surrounding the domain P_δ moving with the failure zone center line velocity V^ϵ.**

Change of internal energy inside P_δ

$$\frac{d}{dt}\int_{\mathcal{P}\delta(t)} T^\epsilon + W^\epsilon \, d\boldsymbol{x} = I^\epsilon(\Gamma_\delta(t))$$

with

$$I^\epsilon(\Gamma_\delta(t)) = \int_{\Gamma_\delta(t)} (T^\epsilon + W^\epsilon) V^\epsilon \boldsymbol{e}^1 \cdot \boldsymbol{n} \, ds - E^\epsilon(\Gamma_\delta(t)),$$

and

$$E^\epsilon(\Gamma_\delta(t)) = \int_{A_\delta(t)} \int_{\mathcal{H}_\epsilon(\boldsymbol{x}) \cap \mathcal{P}_\delta(t)} \partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^\epsilon)) \boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}} \cdot (\dot{\boldsymbol{u}}^\epsilon(\boldsymbol{x}) + \dot{\boldsymbol{u}}^\epsilon(\boldsymbol{y})) \, d\boldsymbol{y} d\boldsymbol{x},$$

where $\boldsymbol{n}$ is the unit normal pointing out of the domain $\mathcal{P}_\delta(t)$ and $A_\delta(t)$ is the part of D exterior to $\mathcal{P}_\delta(t)$.

No Mott hypothesis needed to include kinetic energy - instead a formula for change in internal energy in a domain **containing** the crack.

- Calculation gives that the flux of the stress power density into the domain is related to fracture toughness by

$$\int_{\Gamma_\delta(t)} W^\epsilon V^\epsilon \mathbf{e}^1 \cdot \boldsymbol{n}\, ds = -\mathcal{G}_c V^\epsilon(t) + O(\delta). \tag{14}$$

  Jha & Lipton, arXiv.org ArXiv:1908.07589, preprint 2020.

- For remote loads far away from the crack tip the rate of change in internal energy as $\epsilon \to 0$ of a neighborhood of the crack tip is

$$\lim_{\epsilon \to 0} \frac{d}{dt} \int_{\mathcal{P}_\delta(t)} T^\epsilon + W^\epsilon\, d\boldsymbol{x} = \frac{1+\nu}{E} \frac{V^3}{c_s^2 D} \alpha_t K_I^2(t) - V\mathcal{G}_c + O(\delta), \tag{15}$$

  Here V is crack tip velocity

Kinetic relation for crack tip velocity

- Energy balance:

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\frac{d}{dt}\int_{\mathcal{P}^{\epsilon}_{\delta}(t)} T^{\epsilon}+W^{\epsilon}\,d\boldsymbol{x}=0 \tag{16}$$

we get a Kinetic relation for crack tip motion Jha & Lipton, arXiv.org ArXiv:1908.07589.

- This kinetic relation is **exactly the one given by R. Clifton, L.B. Fruend, J.R. Willis,** given by,

$$\mathcal{G}_c=\frac{1+\nu}{E}\frac{V^2}{c_s^2 D}\alpha_t K_I^2(t), \tag{17}$$

- More generally we have the formula

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{d}{dt}\int_{\mathcal{P}_{\delta}(t)} T^{\epsilon}+W^{\epsilon}\,d\boldsymbol{x}=0\\ &=\int_{\Gamma_\delta}\mathbb{C}\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}\cdot\dot{\boldsymbol{u}}^0\,ds-\mathcal{G}_c V(t)+O(\delta).\end{aligned} \tag{18}$$

Numerical Simulations

- The bond potential $f(p) = c(1 - e^{-\beta p})$ and $J(q) = 1 - q$ where c and β are positive constants.
- These constants are determined so that the bulk modulus k and the critical energy release rate $\mathcal{G}_c$ are k=25GPa, $\nu = 0.245$, $\mathcal{G}_c$=500Jm^{-2}.
- The density is ρ=1200kg-m^{-3}.
- The maximum in the bond force curve occurs at $\mathcal{S}_c = 1/\sqrt{2\beta|y - x|}$.
- The hydrostatic potential is $g(\rho) = C\rho^2/2$
- Here the constants C, c, β are calibrated to k and $\mathcal{G}_c$ as shown above.

First example

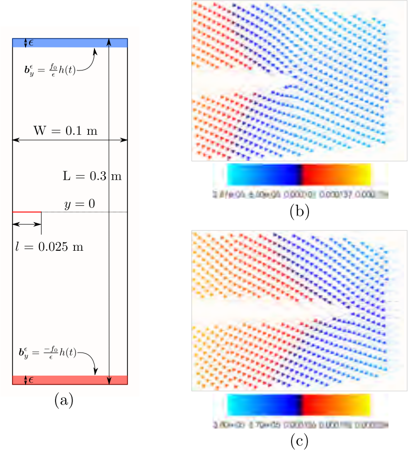


Figure: **Displacement in the domain at time $t = 460$ μs and $t = 520$ μs for horizon $\epsilon = 0.625$ mm.**

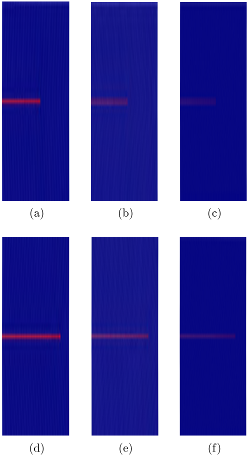


Figure: Failure zone widths (red) $2\epsilon + 2h$. (a), (b), (c) correspond to $FZ^{\epsilon}(t)$ at $t = 460\,\mu$s for $\epsilon = 2.5, 1.25, 0.625$ mm. (d), (e), (f) correspond to $FZ^{\epsilon}(t)$ at $t = 520\,\mu$s for $\epsilon = 2.5, 1.25, 0.625$ mm.

Closer look

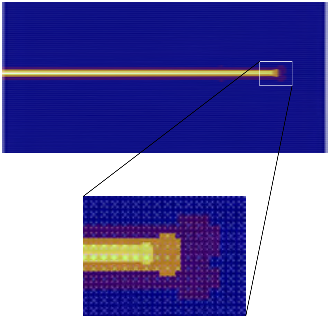


Figure: Top: Failure zone $FZ^{\epsilon}(t)$ for $\epsilon = 2.5, 1.25, 0.625$ mm at time $t = 520\,\mu$s on top of each other. Red, light yellow, and light blue color is used for FZ^{ϵ} of horizon $2.5, 1.25, 0.625$ mm respectively. Bottom: Zoomed in near the crack center line tip.

Crack length vs time

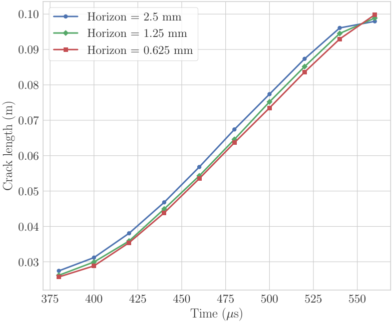


Figure: The crack center line length is plotted as a function of time for three different horizons.

Interaction of crack with circular void

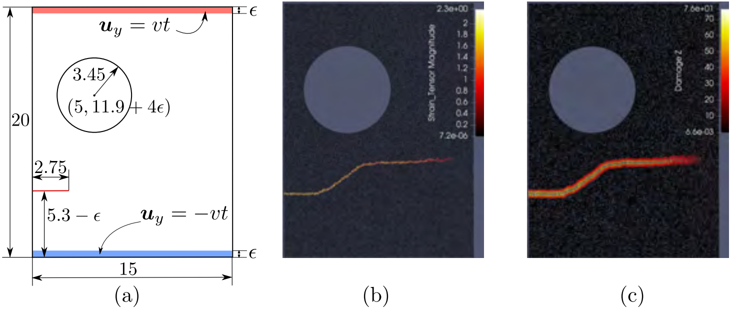
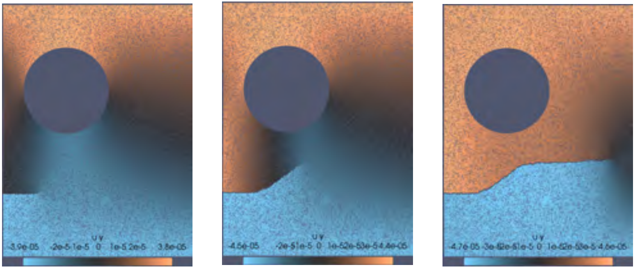


Figure: (a) Setup. Here $\epsilon = 0.32$ mm and $v = 250$ mm/s. (b) Plot of magnitude of the strain at $t = 178\,\mu$s. (c) Plot of the failure zone at $t = 178\,\mu$s. Failure zone width shown coincides with $2\epsilon + 2h$, h is mesh size for numerical discretization.

y component of displacement



(a) (b) (c)

Figure: y component of displacement at different time 150 μs (a), 174.4 μs (b), 178 μs (c).

Stress power density flux + Kinetic flux vs $V\mathcal{G}_c$

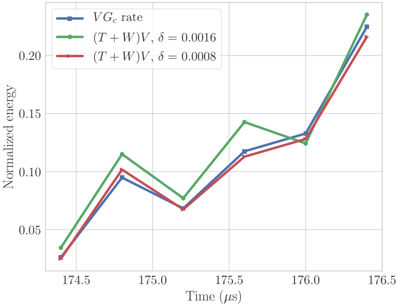


Figure: Stress power density flux $\int_{\mathcal{P}_\delta} W \times V \, ds$ and LEFM energy rate VG_c. Here energy rates are divided by $c_S G_c$, where shear wave speed of the material 3 is $c_s = 5715.47$ m/s and $G_c = 10204.097$ J/m^2. The result shows that agreement is better when contour is small as is shown by calculation.