Evolution equation for the damage variable rewritten to ensure $\dot{\mathsf{d}} \geq 0$

- **"Driving energy"** per unit volume for damage growth:

$$\psi_0 \overset{\text{def}}{=} \underbrace{J^c \left[\underbrace{\left(G|\mathbf{E}^e|^2 + \frac{1}{2}\left(K - \frac{2}{3}G\right)(\text{tr}\,\mathbf{E}^e)^2 \right)}_{\text{elastic energy}} + \underbrace{(1-\varkappa)S^c\epsilon^c}_{\text{craze disordering energy}} \right]}_{\text{“driving energy'' for damage growth}}.$$

- With ϵ^c_{cr} and ψ_{cr}, respectively, representing **threshold values of the craze strain and an energy for initiation of damage**, let (cf. Miehe et al., 2010)

$$\mathcal{H} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } \epsilon^c < \epsilon^c_{\text{cr}} \\ \max\limits_{s\in[0,t]} \left[\langle \psi_0(s) - \psi_{\text{cr}} \rangle\right] & \text{otherwise} \end{cases},$$

- Then the evolution of d is taken to be governed by the partial differential equation,

$$\boxed{\zeta\,\dot{\mathsf{d}} = \big\langle 2(1-\mathsf{d})\,\mathcal{H} - 2\psi_*(\mathsf{d} - \ell^2\,\Delta\mathsf{d})\big\rangle\,,}$$

where $\zeta > 0$ is a (small) **viscous regularization parameter**.

Governing pdes and bcs

- In the absence of body forces and neglecting inertia,

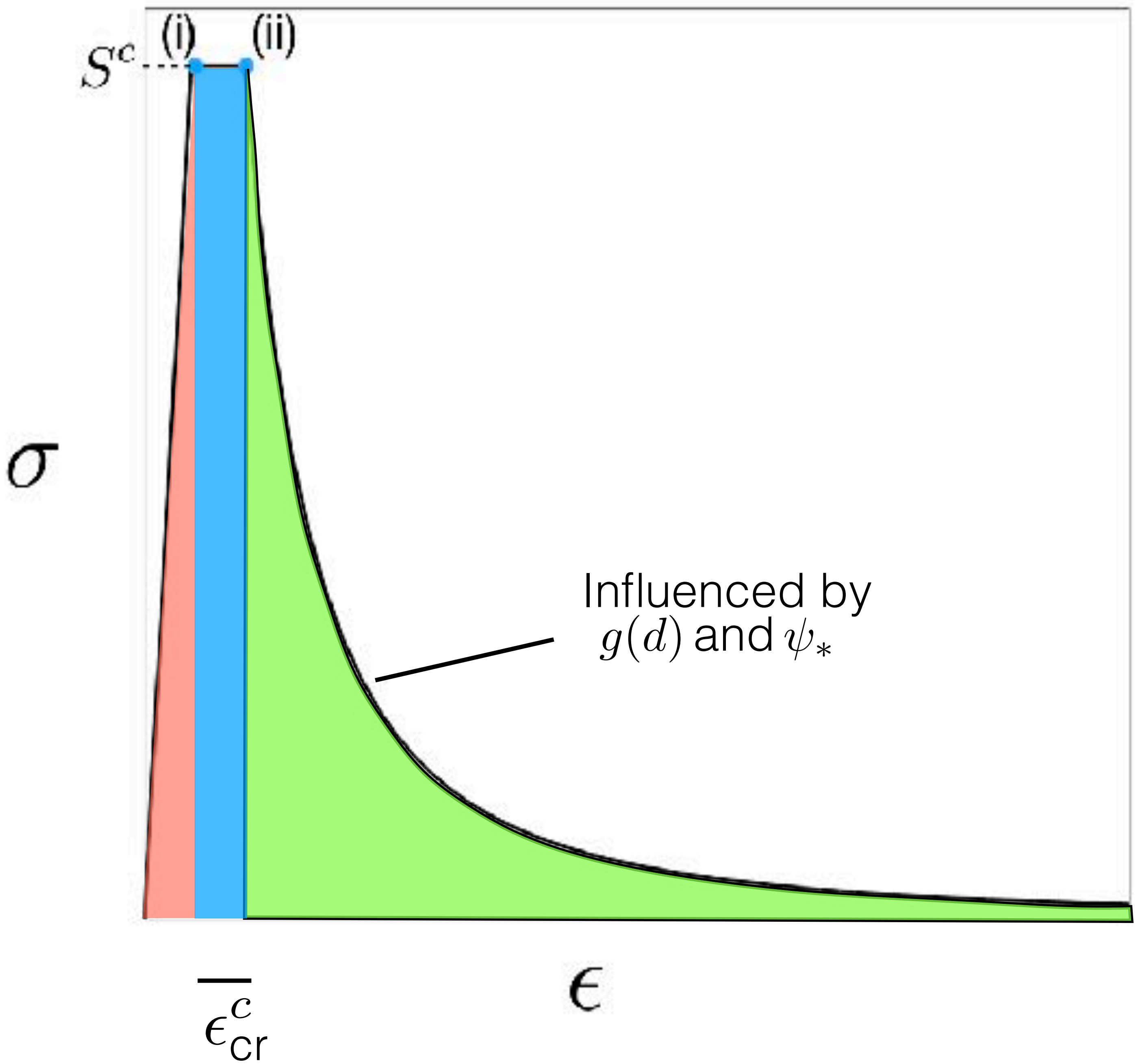
Equilibrium equation
$$\begin{cases} \mathrm{Div}\, \mathbf{T}_{\mathrm{R}} = \mathbf{0} \\ \mathbf{u} = \breve{\mathbf{u}} \quad \text{on } \mathcal{S}_{\mathbf{u}} \times [0, T] \\ \mathbf{T}_{\mathrm{R}} \mathbf{n}_{\mathrm{R}} = \breve{\mathbf{t}}_{\mathrm{R}} \quad \text{on } \mathcal{S}_{\mathbf{t}_{\mathrm{R}}} \times [0, T] \end{cases}$$

Damage evolution equation
$$\begin{cases} \zeta\, \dot{\mathrm{d}} = 2(1 - \mathrm{d})\, \mathcal{H} - 2\psi_*(\mathrm{d} - \ell^2\, \Delta \mathrm{d}) \\ \mathrm{d} = 0 \quad \text{on } \mathcal{S}_{\mathrm{d}} \times [0, T] \\ (\nabla \mathrm{d}) \cdot \mathbf{n}_{\mathrm{R}} = 0 \quad \text{on } \mathcal{S}_{\boldsymbol{\xi}} \times [0, T] \end{cases}$$

Initial conditions: $\mathbf{u}(\mathbf{X}, 0) = \mathbf{0}$ and $\mathrm{d}(\mathbf{X}, 0) = 0$ in B.

- Implemented in Abaqus as a UEL

Single Element response



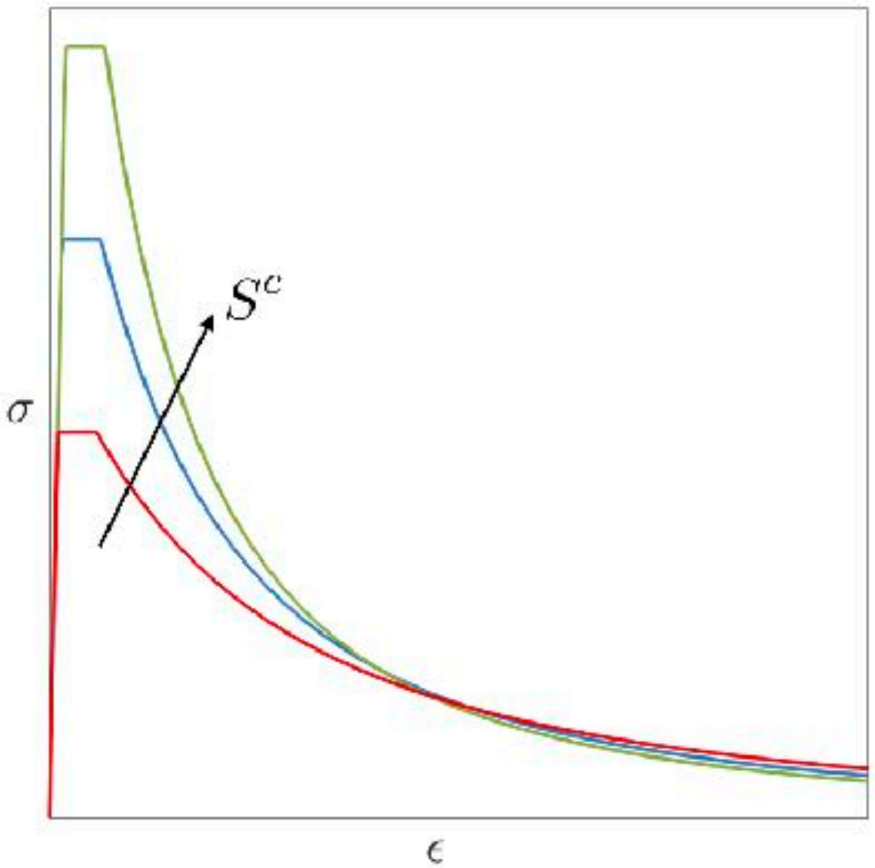
- As an estimate for ψ_{cr}, which is an energy barrier for damage initiation, we take

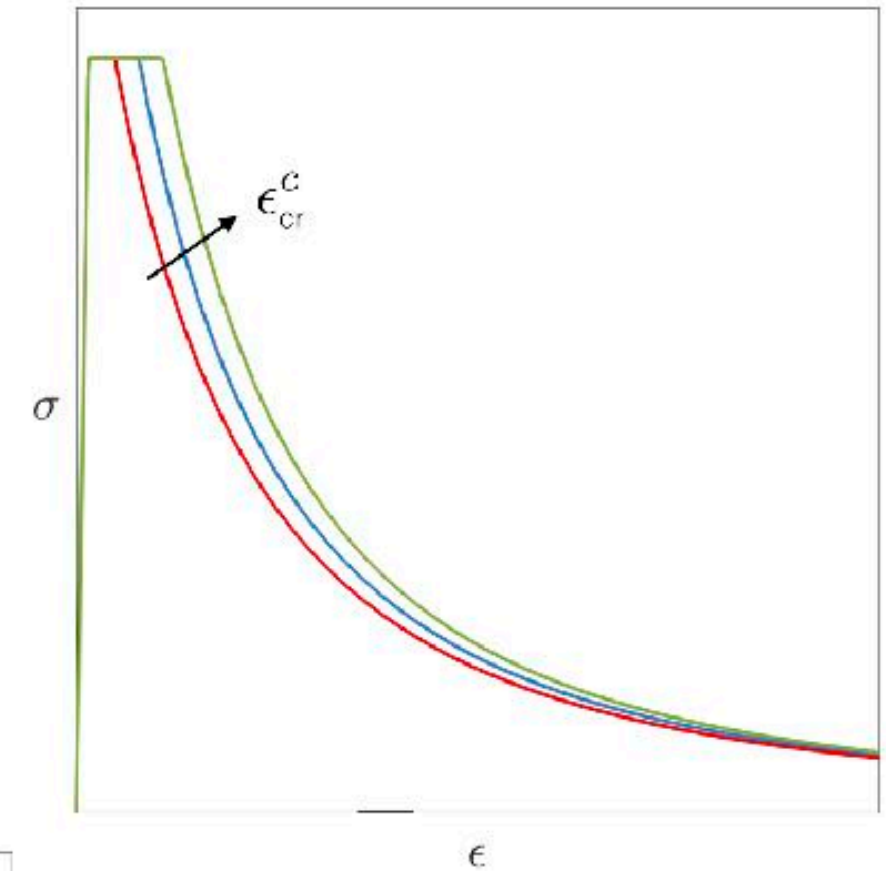
$$\psi_{\text{cr}} = \frac{1}{2}\frac{S^{c\,2}}{E} + (1-\varkappa)S^c\,\epsilon^c_{\text{cr}}.$$

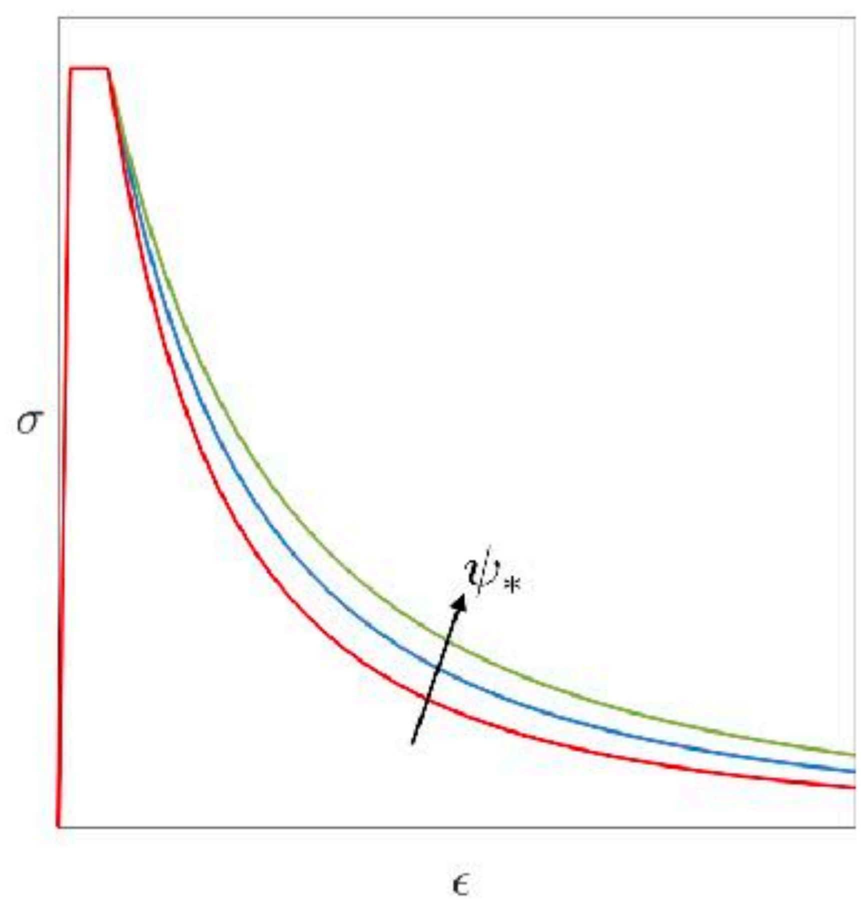
- We take

$$\epsilon^c_{\text{cr}} \propto \frac{S^c}{E}$$

Single Element response







Material parameters

1. The elastic shear and bulk moduli, G and K. Or equivalently, the Young's modulus and Poisson's ratio E and ν.
2. The craze strength parameter, S^c.
3. A fractional parameter $\varkappa \in (0,1)$, where $(1-\varkappa)$ controls the amount of energy per unit volume stored due to crazing. We take $\varkappa \approx 0.7$.
4. The parameters ϵ^c_{cr} and ψ_{cr} represent a critical craze strain and energy that must be reached for damage to initiate.
5. The parameter ψ_* represents a contribution to the energy dissipated as damage grows from zero to unity.
6. The length scale parameter ℓ controls the spread of the diffuse damage zone.
7. The parameter ζ is viscous regularization parameter for the evolution of damage.

Some distinguishing features of our theory

- Craze inelasticity is allowed to occur only when the maximum principal stress is positive. There is no need in our theory to decompose the free energy into "positive" and "negative" parts to avoid cracking under "compressive" states of strain, as done in most existing theories for phase-field modeling of fracture.

- Our theory contains a material parameter ψ_{cr} which sets a level of energy that must be exceeded before damage initiates.

- The term $\psi_* \ell^2 |\nabla \mathrm{d}|^2$ is ***energetic*** in our theory since it appears in the free energy, and therefore the term $2\psi_* \ell^2 \Delta \mathrm{d}$ which appears in the evolution equation for $\mathtt{d}$ is also ***energetic*** and not ***dissipative*** — as is commonly assumed in the literature by many including Miehe and co-workers.

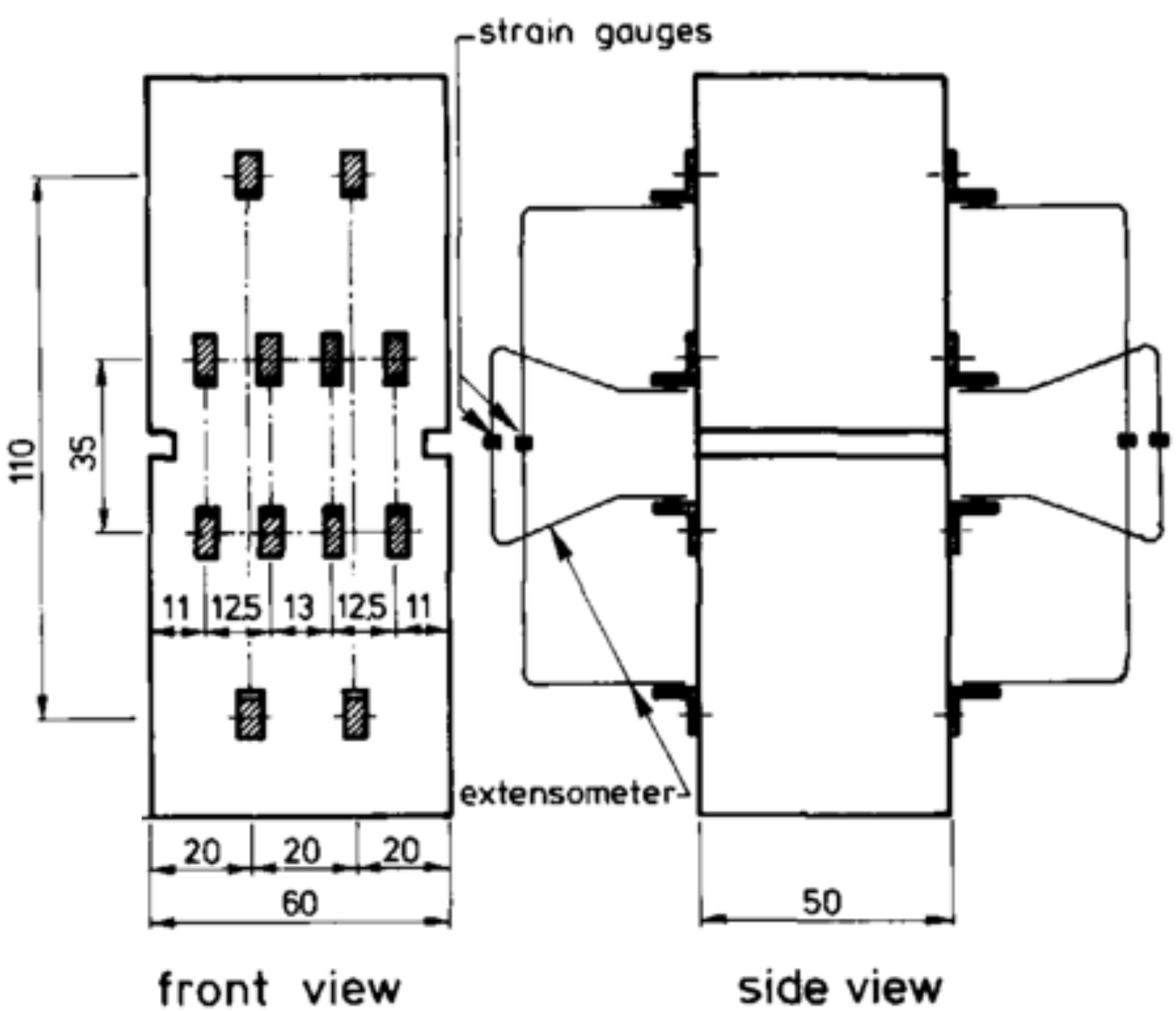
- The value of the strength parameter S^c in our theory is not related to the elastic Young's modulus E, a toughness G_c, and the length scale ℓ — as in most existing phase-field theories of fracture. In our theory the material strength is controlled directly by S^c.

- The parameter ℓ is a suitable gradient regularization parameter which may be ***independently*** prescribed based on physical considerations of the microstructure and computational-tractability.

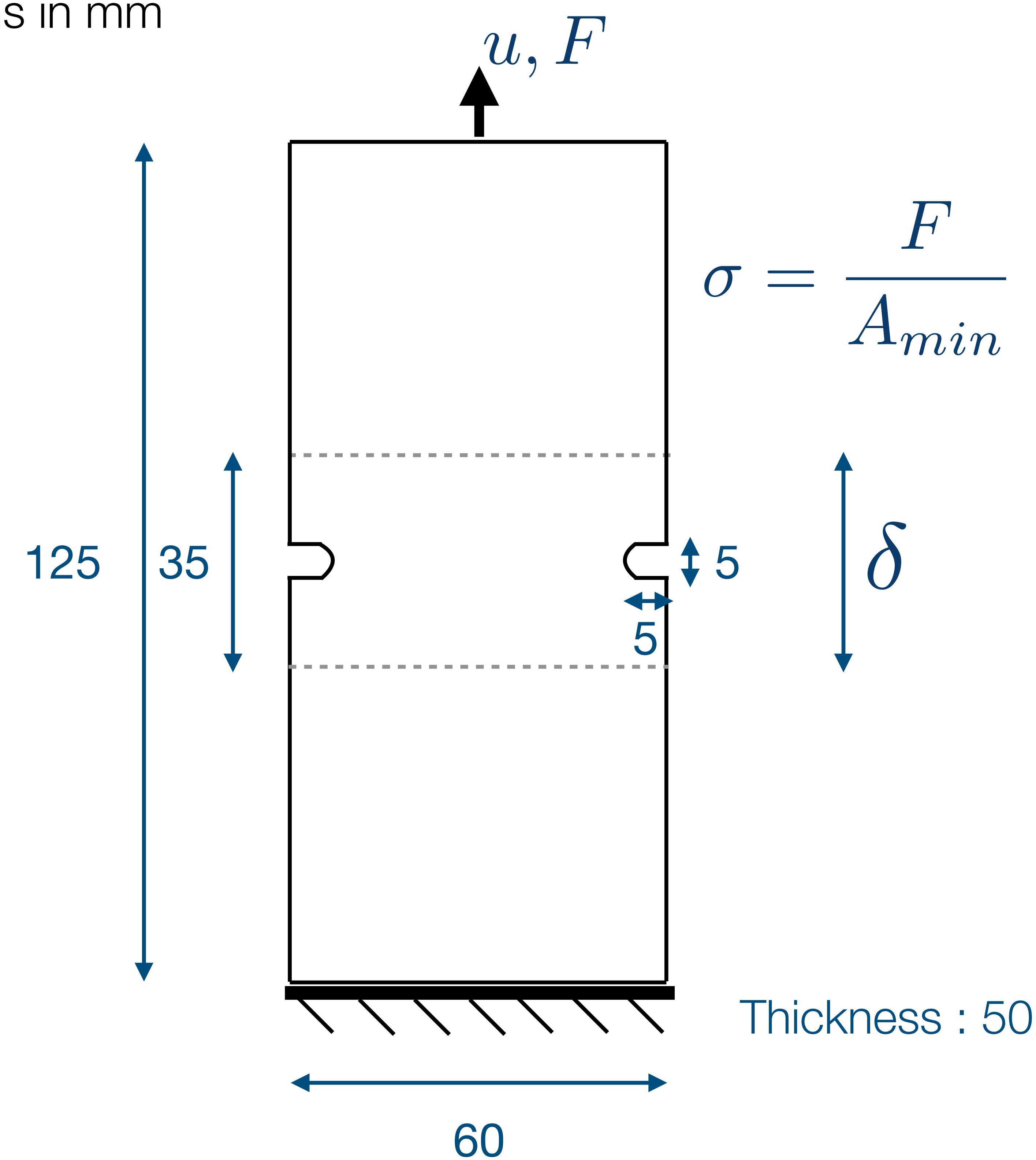
Simulation results for concrete

Direct Tension Test

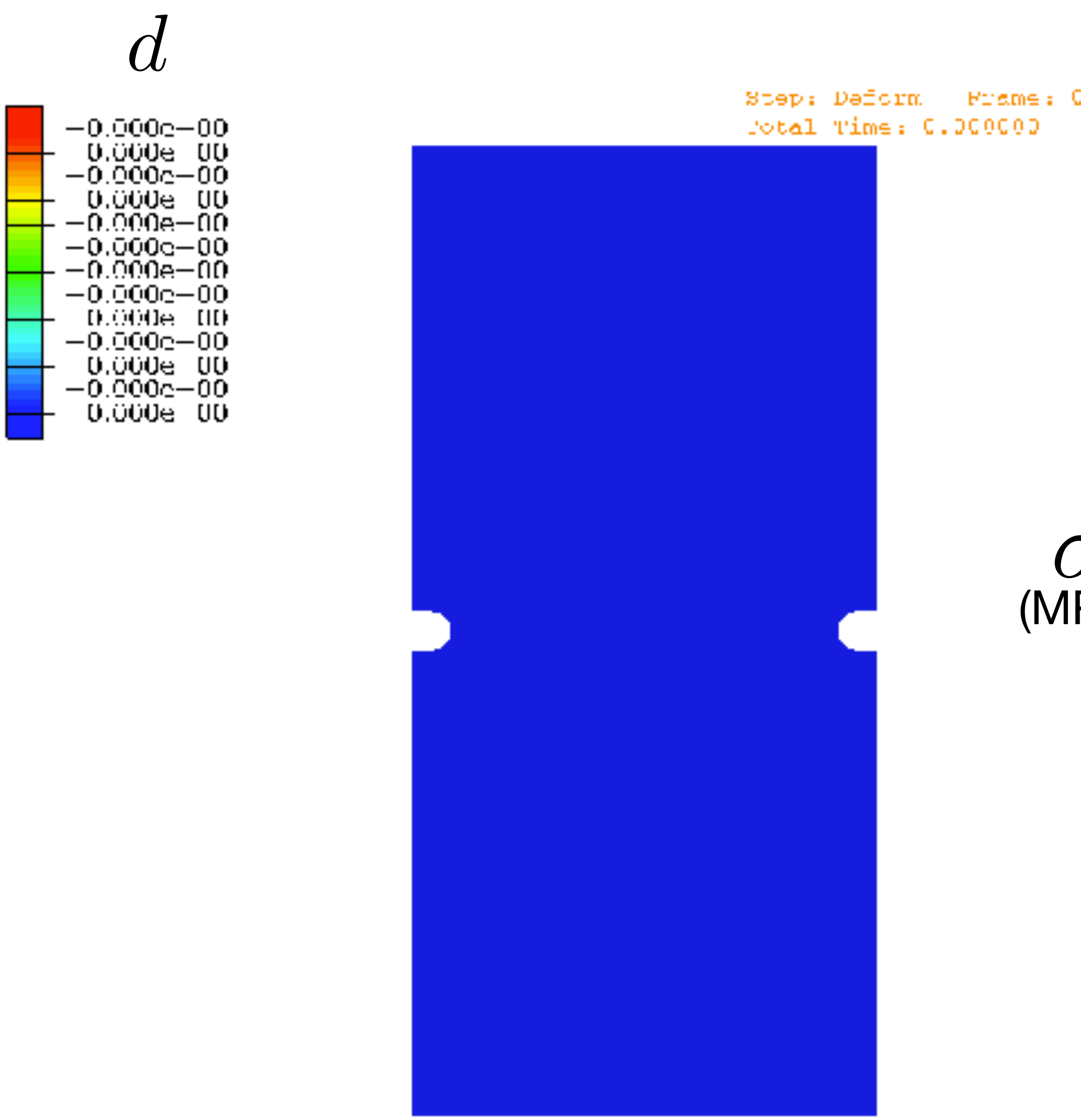
Dimensions in mm



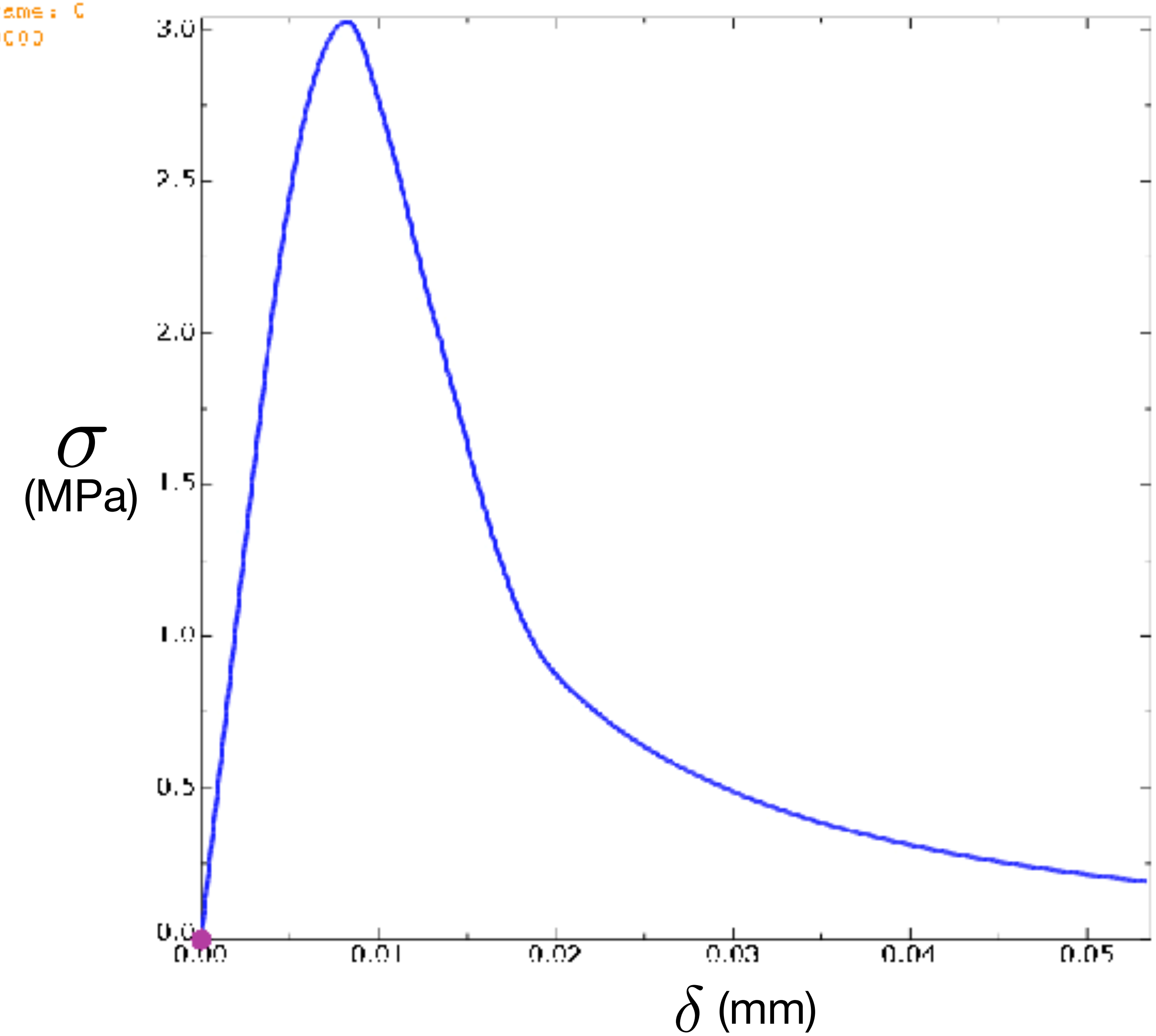
(Hordijk,1991)



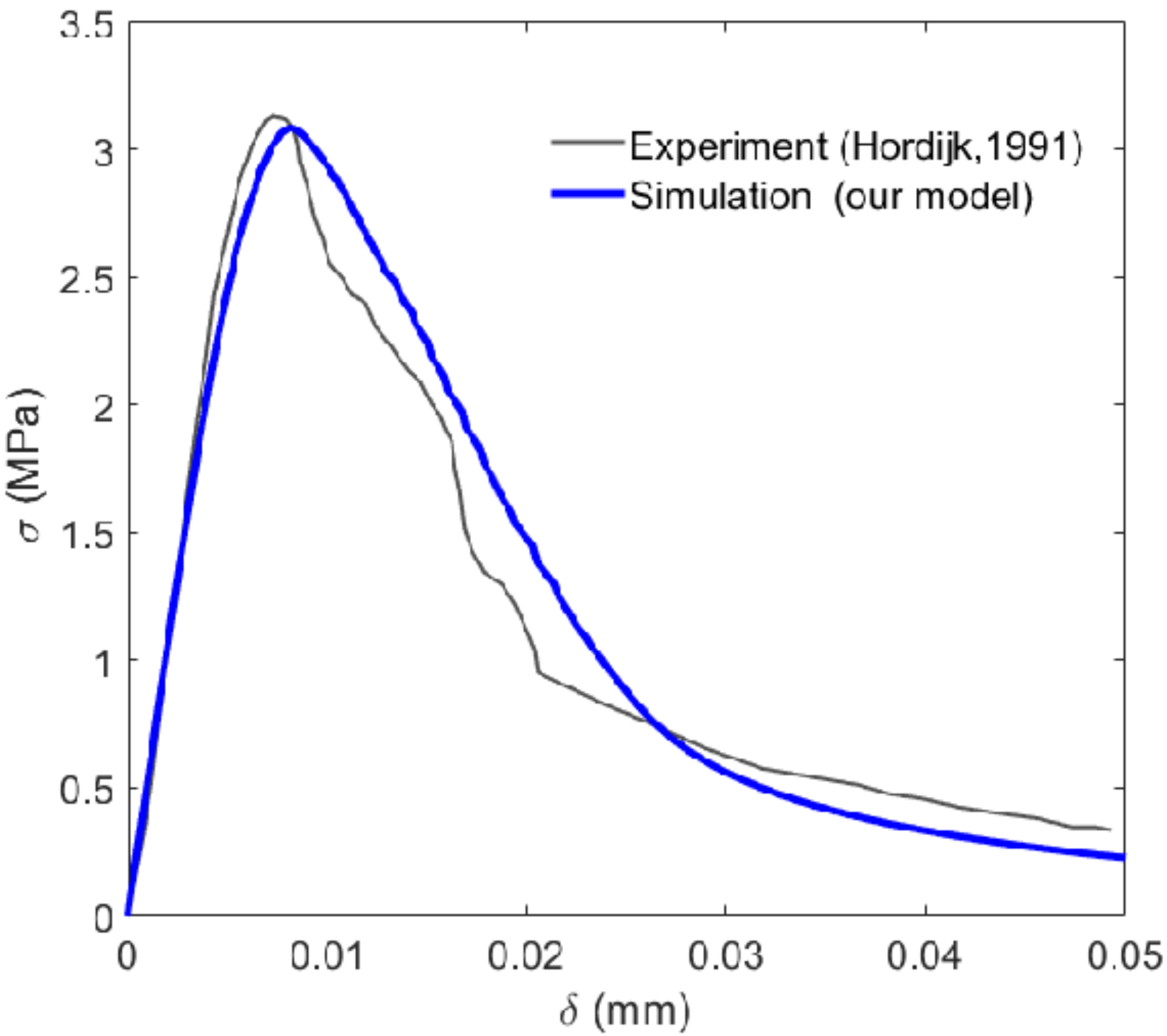
Simulation



Displacement magnified 50x



Direct Tension Test



Material Parameters

E	18	GPa
ν	0.2	-
S^c	3.2	MPa
$\varkappa$	0.7	-
ϵ^c_{crit}	5.3×10^{-4}	-
ψ_*	4.0	kJ/m ^{3}
ℓ	2	mm
ζ	40	kPa-s