Virtual velocities

- Consider the fields

$$\dot{\boldsymbol{\chi}},\ \dot{\mathbf{F}}^e,\ \dot{\epsilon}^c,\ \dot{\mathsf{d}}$$

as **virtual velocities** and denote the virtual fields by

$$\mathcal{V} = (\tilde{\boldsymbol{\chi}}, \tilde{\mathbf{F}}^e, \tilde{\epsilon}^c, \tilde{\mathsf{d}}).$$

We require that they satisfy

$$(\nabla\tilde{\boldsymbol{\chi}})\mathbf{F}^{-1} = \tilde{\mathbf{F}}^e\mathbf{F}^{e-1} + \tilde{\epsilon}^c\mathbf{F}^e\,\mathbf{N}^c\,\mathbf{F}^{e-1}, \qquad \tilde{\mathsf{d}} \geq 0.$$

- We refer to a macroscopic virtual field $\mathcal{V}$ as **rigid** if it satisfies

$$(\nabla\tilde{\boldsymbol{\chi}}) = \boldsymbol{\Omega}\mathbf{F},$$

with $\boldsymbol{\Omega}$ a spatially constant skew tensor, together with

$$\tilde{\mathbf{F}}^e = \boldsymbol{\Omega}\mathbf{F}^e, \quad \tilde{\epsilon}^c = 0, \quad \tilde{\mathsf{d}} = 0.$$

Principle of virtual power

$$\mathcal{W}_{\text{ext}}(\mathrm{P},\mathcal{V}) = \int_{\partial \mathrm{P}} \mathbf{t}_{\mathrm{R}}(\mathbf{n}_{\mathrm{R}}) \cdot \tilde{\boldsymbol{\chi}}\, da_{\mathrm{R}} + \int_{\mathrm{P}} \mathbf{b}_{\mathrm{R}} \cdot \tilde{\boldsymbol{\chi}}\, dv_{\mathrm{R}} + \int_{\partial \mathrm{P}} \xi(\mathbf{n}_{\mathrm{R}})\, \tilde{\mathrm{d}}\, da_{\mathrm{R}},$$

$$\mathcal{W}_{\text{int}}(\mathrm{P},\mathcal{V}) = \int_{\mathrm{P}} \left(\mathbf{S}^e : \tilde{\mathbf{F}}^e + \pi \tilde{\epsilon}^c + \varpi \tilde{\mathrm{d}} + \boldsymbol{\xi} \cdot \nabla \tilde{\mathrm{d}} \right) dv_{\mathrm{R}},$$

- The **principle of virtual power** consists of two basic requirements:

(V1) Given any part P,

$$\mathcal{W}_{\text{ext}}(\mathrm{P},\mathcal{V}) = \mathcal{W}_{\text{int}}(\mathrm{P},\mathcal{V}) \quad \text{for all generalized virtual velocities } \mathcal{V}.$$

(V2) Given any part P and a ***rigid*** virtual velocity $\mathcal{V}$,

$$\mathcal{W}_{\text{int}}(\mathrm{P},\mathcal{V}) = 0 \quad \text{whenever } \mathcal{V} \text{ is a rigid macroscopic virtual velocity.}$$

Consequences of principle of virtual power

$$\mathcal{W}_{\text{int}}(\mathrm{P},\mathcal{V}) = \int_{\mathrm{P}} \Big(\mathbf{S}^e : \tilde{\mathbf{F}}^e + \pi\tilde{\epsilon}^c + \varpi\tilde{\mathsf{d}} + \boldsymbol{\xi}\cdot\nabla\tilde{\mathsf{d}} \Big)\, dv_{\mathrm{R}}$$

(a) **Macroscopic force and moment balances**: The stress

$$\mathbf{T}_{\mathrm{R}} \overset{\text{def}}{=} \mathbf{S}^e\mathbf{F}^{c-\top} \quad \text{satisfies} \quad \mathbf{T}_{\mathrm{R}}\mathbf{F}^\top = \mathbf{F}\mathbf{T}_{\mathrm{R}}^\top,$$

and satisfies a **macroscopic force balance** and a **macroscopic traction condition**,

$$\mathrm{Div}\,\mathbf{T}_{\mathrm{R}} + \mathbf{b}_{0\mathrm{R}} = \mathbf{0} \qquad \text{and} \qquad \mathbf{t}_{\mathrm{R}}(\mathbf{n}_{\mathrm{R}}) = \mathbf{T}_{\mathrm{R}}\mathbf{n}_{\mathrm{R}},$$

so $\mathbf{T}_{\mathrm{R}}$ represents the classical Piola stress.

– The Cauchy stress $\mathbf{T}$:

$$\mathbf{T} = J^{-1}\mathbf{T}_{\mathrm{R}}\mathbf{F}^\top \quad \in \text{sym}.$$

– The elastic second Piola stress:

$$\mathbf{T}^e \overset{\text{def}}{=} J^e\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^{e-\top} \quad \in \text{sym},$$

– The Mandel stress:

$$\mathbf{M}^e \overset{\text{def}}{=} \mathbf{C}^e\mathbf{T}^e = J^e\mathbf{F}^{e\top}\mathbf{T}\mathbf{F}^{e-\top}.$$

Consequences of principle of virtual power

$$\mathcal{W}_{\mathrm{ext}}(\mathrm{P},\mathcal{V}) = \int_{\partial \mathrm{P}} \mathbf{t}_{\mathrm{R}}(\mathbf{n}_{\mathrm{R}}) \cdot \tilde{\boldsymbol{\chi}}\, da_{\mathrm{R}} + \int_{\mathrm{P}} \mathbf{b}_{\mathrm{R}} \cdot \tilde{\boldsymbol{\chi}}\, dv_{\mathrm{R}} + \int_{\partial \mathrm{P}} \xi(\mathbf{n}_{\mathrm{R}})\, \tilde{\mathrm{d}}\, da_{\mathrm{R}},$$

$$\mathcal{W}_{\mathrm{int}}(\mathrm{P},\mathcal{V}) = \int_{\mathrm{P}} \Big(\mathbf{S}^e : \tilde{\mathbf{F}}^e + \pi \tilde{\epsilon}^c + \varpi \tilde{\mathrm{d}} + \boldsymbol{\xi} \cdot \nabla \tilde{\mathrm{d}}\Big)\, dv_{\mathrm{R}}$$

(b) A first microscopic force balance for crazing, ϵ^c:

Let

$$\sigma \stackrel{\mathrm{def}}{=} J^c \mathbf{M}^e : \mathbf{N}^c\,.$$

denote a ***resolved tensile stress***. Then

$$\boxed{\sigma = \pi,}$$

(c) A second microscopic force balance and traction condition for the damage d and its gradient ∇d:

$$\boxed{\mathrm{Div}\,\boldsymbol{\xi} - \varpi = 0, \qquad \text{and} \qquad \xi(\mathbf{n}_{\mathrm{R}}) = \boldsymbol{\xi}\cdot\mathbf{n}_{\mathrm{R}}.}$$

Actual external and internal expenditures of power

- The actual external expenditure of power:

$$\mathcal{W}_{\text{ext}}(\mathrm{P}) = \int_{\partial \mathrm{P}} (\mathbf{T}_{\mathrm{R}}\mathbf{n}_{\mathrm{R}}) \cdot \dot{\boldsymbol{\chi}}\, da_{\mathrm{R}} + \int_{\mathrm{P}} \mathbf{b}_{\mathrm{R}} \cdot \dot{\boldsymbol{\chi}}\, dv_{\mathrm{R}} + \int_{\partial \mathrm{P}} (\boldsymbol{\xi} \cdot \mathbf{n}_{\mathrm{R}})\, \dot{\mathrm{d}}\, da_{\mathrm{R}}.$$

- Since the stress power $\mathbf{S}^e : \dot{\mathbf{F}}^e$ may be alternatively written as,

$$\mathbf{S}^e : \dot{\mathbf{F}}^e = \frac{1}{2} J^c \mathbf{T}^e : \dot{\mathbf{C}}^e,$$

the actual internal expenditure of power may be written as

$$\mathcal{W}_{\text{int}}(\mathrm{P}) = \int_{\mathrm{P}} \Big(\frac{1}{2} J^c \mathbf{T}^e : \dot{\mathbf{C}}^e + \pi \dot{\epsilon}^c + \varpi \dot{\mathrm{d}} + \boldsymbol{\xi} \cdot \nabla \dot{\mathrm{d}} \Big)\, dv_{\mathrm{R}}.$$

Free-energy imbalance under isothermal conditions

- Under isothermal conditions the free-energy imbalance is the statement:

$$\overline{\int_{\mathrm{P}} \psi_{\mathrm{R}}\, dv_{\mathrm{R}}}^{\,\cdot} \le \mathcal{W}_{\mathrm{ext}}(\mathrm{P}) = \mathcal{W}_{\mathrm{int}}(\mathrm{P}),$$

$$\Downarrow$$

$$\int_{\mathrm{P}} \left[\dot{\psi}_{\mathrm{R}} - \left(\frac{1}{2} J^c \mathbf{T}^e : \dot{\mathbf{C}}^e + \pi \dot{\epsilon}^c + \varpi \dot{\mathsf{d}} + \boldsymbol{\xi} \cdot \nabla \dot{\mathsf{d}} \right) \right] dv_{\mathrm{R}} \le 0,$$

$$\Downarrow$$

$$\boxed{\dot{\psi}_{\mathrm{R}} - \frac{1}{2} J^c \mathbf{T}^e : \dot{\mathbf{C}}^e - \pi \dot{\epsilon}^c - \varpi \dot{\mathsf{d}} - \boldsymbol{\xi} \cdot \nabla \dot{\mathsf{d}} \le 0\,.}$$

Constitutive equations

(1) Free energy:

$$\psi_{\mathrm{R}} = g(\mathrm{d})\psi_0 + \psi_* \ell^2 \,|\nabla \mathrm{d}|^2, \quad \text{with}$$

$$\psi_0 = \underbrace{J^c \left[\underbrace{G|\mathbf{E}^e|^2 + \frac{1}{2}\left(K - \frac{2}{3}G\right)(\mathrm{tr}\,\mathbf{E}^e)^2}_{\text{elastic energy}} + \underbrace{(1-\varkappa)\,S^c\epsilon^c}_{\text{defect energy}} \right]}_{\text{“undamaged” energy}}.$$

(i) $G > 0$ and $K > 0$ are the shear and bulk moduli, respectively.

(ii) $S^c\epsilon^c$ represents an inelastic work expended due to crazing and $\varkappa$ a fraction in the range $\varkappa \in (0,1)$. We assume that the fraction $\varkappa S^c\epsilon^c$ is dissipated, while the balance $(1-\varkappa)S^c\epsilon^c$ is stored in the material due to craze-disordering,

$$S^c\epsilon^c = \underbrace{\varkappa S^c\epsilon^c}_{\text{energy dissipated due to crazing}} + \underbrace{(1-\varkappa)S^c\epsilon^c}_{\text{defect energy stored due to craze disordering}}.$$

(iii) $g(\mathrm{d}) = (1-\mathrm{d})^2$ is a monotonically decreasing **degradation function**.

(iv) The parameter ψ_* is an energy per unit volume associated with the evolution of damage.

(v) The parameter $\ell > 0$ is a length scale that controls the spread of the diffuse damage zone.

Constitutive equations

(2) Mandel stress:

$$\mathbf{M}^e = J^{c-1}\left(\frac{\partial \psi_{\mathsf{R}}}{\partial \mathbf{E}^e}\right) = g(\mathsf{d})\left[2G\mathbf{E}_0^e + K(\mathrm{tr}\,\mathbf{E}^e)\mathbf{1}\right],$$

which is symmetric.

(i) The spectral decomposition of the Mandel stress is

$$\mathbf{M}^e = \sum_{i=1}^{3} \sigma_i\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i \quad \text{with} \quad \sigma_1 \geq \sigma_2 \geq \sigma_3,$$

where $\{\sigma_i | i = 1, 2, 3\}$ are the principal values and $\{\hat{\mathbf{e}}_i | i = 1, 2, 3\}$ are the principal directions of $\mathbf{M}^e$.

(ii) Craze inelasticity will be taken to occur in the maximum principal stress direction $\hat{\mathbf{e}}_1$.

(3) Evolution equation for $\mathbf{F}^c$

$$\dot{\mathbf{F}}^c = \mathbf{D}^c\mathbf{F}^c \quad \text{with} \quad \mathbf{F}^c(\mathbf{X},0) = \mathbf{1}, \quad \text{where}$$

$$\mathbf{D}^c = \dot{\epsilon}^c\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1, \quad \text{and}$$

$$\dot{\epsilon}^c = \begin{cases} > 0 \text{ possible} & \text{if } \sigma_1 > 0 \quad \text{and} \quad \sigma_{\mathrm{M}} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The **craze strain** is defined by

$$\epsilon^c(t) \stackrel{\text{def}}{=} \int_0^t \dot{\epsilon}^c(s)\, ds.$$

- With $S^c > 0$ denoting a **resistance to craze flow**, we introduce a **yield condition**:

$$f \stackrel{\text{def}}{=} \sigma_1 - g(\mathsf{d})S^c \leq 0.$$

- **Loading-unloading conditions** may be expressed in the Kuhn-Tucker form,

$$\dot{\epsilon}^c \geq 0, \quad f \leq 0, \quad \dot{\epsilon}^c f = 0,$$

- **Consistency condition**:

$$\dot{\epsilon}^c \dot{f} = 0 \quad \text{when} \quad f = 0.$$

The consistency condition serves to determine $\dot{\epsilon}^c$ whenever it is not zero.

Constitutive equations

(3) Evolution equation for the damage variable d:

- Const. eqns. for the microstress ϖ and $\boldsymbol{\xi}$:

$$\varpi = \underbrace{-2(1-\mathsf{d})\,\underbrace{J^c\left(\tilde{\psi}^e(\mathcal{I}_{\mathbf{E}^e}) + \tilde{\psi}^c(\epsilon^c)\right)}_{\psi_0 \text{ undamaged energy}}}_{\text{energetic}=\dfrac{\partial \psi_{\mathsf{R}}}{\partial \mathsf{d}}} + \underbrace{2(1-\mathsf{d})\psi_{\mathsf{cr}} + 2\psi_*\mathsf{d} + \zeta\,\dot{\mathsf{d}}}_{\text{dissipative}},$$

$$\boldsymbol{\xi} = \underbrace{2\psi_*\ell^2\nabla\mathsf{d}}_{\text{energetic}=\dfrac{\partial \psi_{\mathsf{R}}}{\partial \nabla\mathsf{d}}}\quad.$$

- Substitution of these const. eqns. in the microforce balance

$$\mathrm{Div}\,\boldsymbol{\xi} - \varpi = 0,$$

yield the following evolution equation for d,

$$\boxed{\zeta\dot{\mathsf{d}} = \left\langle 2(1-\mathsf{d})\,(\psi_0 - \psi_{\mathsf{cr}}) - 2\psi_*(\mathsf{d} - \ell^2\,\Delta\mathsf{d})\right\rangle.}$$